Stanley's Chromatic Polynomial

Recall,

$$X_G(x_1, x_2, ...) := \sum_{\substack{\kappa: V(G) \to \mathbb{N} \\ proper}} \prod_{v \in V(G)} x_{\kappa(v)}$$

Example:

$$\begin{aligned} X_{\bullet\!\!-\!\!\!-\!\!\bullet} &= \widehat{x_1 x_1} + x_1 x_2 + x_1 x_3 + \cdots \\ &\quad + x_2 x_1 + \widehat{x_2 x_2} + x_2 x_3 + \cdots \\ &\quad + x_3 x_1 + x_3 x_2 + \widehat{x_3 x_3} + \cdots \\ &\qquad\qquad \vdots \\ &= (x_1 + x_2 + x_3 + \cdots)^2 - (x_1^2 + x_2^2 + x_3^2 + \cdots) \\ &= p_1^2 - p_2 \end{aligned}$$

where $p_m := \sum_{i=1}^{\infty} x_i^m$.

Elementary Symmetric Functions

$$e_k := \sum_{1 \le j_1 < j_2 < \cdots < j_k} x_{j_1} \cdots x_{j^k}$$

Back to our example:

$$\begin{aligned} X_{\bullet\!\!-\!\!\!-\!\!\bullet} &= \widehat{x_1 x_1} + x_1 x_2 + x_1 x_3 + \cdots \\ &\quad + x_2 x_1 + \widehat{x_2 x_2} + x_2 x_3 + \cdots \\ &\quad + x_3 x_1 + x_3 x_2 + \widehat{x_3 x_3} + \cdots \\ &\qquad\qquad \vdots \\ &= 2(x_1 x_2 + x_1 x_3 + \cdots + x_2 x_3 + x_2 x_4 + \cdots + x_{n-1} x_n + \cdots \\ &= 2e_2 \end{aligned}$$

Note: $X_{K_k} = k! e_k$

B-symmetric chromatic function

$$Y_G(\cdots, x_{-2}, x_{-1}, x_1, x_2, ...) := \sum_{\substack{\kappa: V(G) \to \mathbb{Z} \setminus \{0\} \\ proper}} \prod_{v \in V(G)} x_{\kappa(v)}$$

Signed power functions: $p_{a,b} := \sum_{i \in \mathbb{Z} \setminus \{0\}} x_i^a x_{-i}^b$. Notice that $p_{a,b} = p_{b,a}$.

Example:

$$Y_{\bullet\!\!\bigcirc_-} = \cdots + x_{-2} + x_{-1} + x_1 + x_2 + \cdots = p_{1,0}$$

Example:

$$
\begin{aligned}
Y_{\bullet \overset{-}{\underset{+}{\frown}} \bullet} &= \sum_{i,j} x_i x_j - \sum_i x_i^2 - \sum_i x_i x_{-i} \\
&= p_{1,0}^2 - p_{2,0} - p_{1,1}
\end{aligned}
$$

Contraction-Deletion

Theorem. *For a weighted signed graph G and positive $e \in E(G)$,*

$$Y_G = Y_{G \backslash e} - Y_{G/e}$$

Example:

$$
\begin{aligned}
Y_{\underset{(1,0)}{\bullet} \overset{-}{\text{—}} \underset{(1,0)}{\bullet} \overset{+}{\text{—}} \underset{(1,0)}{\bullet}} &= Y_{\underset{(1,0)}{\bullet} \overset{-}{\text{—}} \underset{(1,0)}{\bullet} \;\; \underset{(1,0)}{\bullet}} - Y_{\underset{(1,0)}{\bullet} \overset{-}{\text{—}} \underset{(2,0)}{\bullet}} \\
&= Y_{\underset{(1,0)}{\bullet} \overset{+}{\text{—}} \underset{(0,1)}{\bullet} \;\; \underset{(1,0)}{\bullet}} - Y_{\underset{(1,0)}{\bullet} \overset{-}{\text{—}} \underset{(2,0)}{\bullet}} \\
&= Y_{\underset{(1,0)}{\bullet} \overset{+}{\text{—}} \underset{(0,1)}{\bullet} \;\; \underset{(1,0)}{\bullet}} - Y_{\underset{(1,0)}{\bullet} \overset{+}{\text{—}} \underset{(0,2)}{\bullet}} \\
&= Y_{\underset{(1,0)}{\bullet} \; \underset{(0,1)}{\bullet} \; \underset{(1,0)}{\bullet}} - Y_{\underset{(1,1)}{\bullet} \; \underset{(1,0)}{\bullet}} - Y_{\underset{(1,0)}{\bullet} \; \underset{(0,2)}{\bullet}} + Y_{\underset{(1,2)}{\bullet}} \\
&= p_{1,0}^3 - p_{1,1} p_{1,0} - p_{1,0} p_{2,0} - p_{2,1}
\end{aligned}
$$

Motivation

Let $\lambda \vdash d$ denote a partition of d, i.e., $\lambda = (\lambda_1, \lambda_2, \cdots)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \sum \lambda_i = d.$$

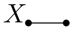
Let $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$.

Theorem (Stanley)**.** *Suppose*

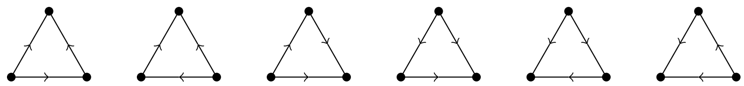
$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$$

is the expansion of X_G in terms of elementary symmetric functions e_λ. Let $a_j(G)$ be the number of acyclic orientations of G with j sinks. Then

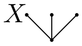
$$a_j(G) = \sum_{\substack{\lambda \vdash d \\ l(\lambda) = j}} c_\lambda .$$

Example: X $= 2e_2$, so this graph has 2 acyclic orientations with exactly one sink: and .

Example: $X_{K_3} = 6e_3$, so a triangle has 6 acyclic orientations with exactly one sink:



Example:

X $= e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}.$

$a_1(G) = 1$, $a_2(G) = 5 - 2 = 3$, $a_3(G) = 1$, and total number of acyclic orientations is $a(G) = 1 + 3 + 1 = 5$.

Goal: Generalize this theorem to B-symmetric chromatic function.